CHAPTER 16
QUADRATIC EQUATIONS IN ONE VARIABLE

The degree of an equation in one variable is the exponent of the highest power to which the variable is raised in that equation. A second-degree equation in one variable is one in which the variable is raised to the second power. A second-degree equation is often called a QUADRATIC EQUATION. The word quadratic is derived from the Latin word quadratus, which means "squared." In a quadratic equation the term of highest degree is the squared term. For example, the following are quadratic equations:

$$x^2 + 3x + 4 = 0$$
$$3m + 4m^2 = 6$$

The terms of degree lower than the second may or may not be present. The possible terms of lower degree than the squared term in a quadratic equation are the first-degree term and the constant term. In the equation

$$3x^2 - 8x - 5 = 0$$

-5 is the coefficient of x^0. If we wished to emphasize the powers of x in this equation, we could write the equation in the form

$$3x^2 - 8x^1 - 5x^0 = 0$$

Examples of quadratic equations in which either the first-degree term or the constant term is missing are:

1. $4x^2 = 16$
2. $y^2 + 16y = 0$
3. $e^2 + 12 = 0$

GENERAL FORM OF A QUADRATIC EQUATION

Any quadratic equation can be arranged in the general form:

$$ax^2 + bx + c = 0$$

If it has more than three terms, some of them will be alike and can be combined, after which the final form will have at most three terms. For example,

$$2x^2 + 3 + 5x - 1 + x^2 = 4 - x^2 - 2x - 3$$

reduces to the simpler form

$$4x^2 + 7x + 1 = 0$$

In this form, it is easy to see that a, the coefficient of x^2, is 4; b, the coefficient of x, is 7; and c, the constant term, is 1.

Sometimes the coefficients of the terms of a quadratic appear as negative numbers, as follows:

$$2x^2 - 3x - 5 = 0$$

This equation can be rewritten in such a way that the connecting signs are all positive, as in the general form. This is illustrated as follows:

$$2x^2 + (-3)x + (-5) = 0$$

In this form, the value of a is seen to be 2, b is -3, and c is -5.

An equation of the form

$$x^2 + 2 = 0$$

has no x term. This can be considered as a case in which a is 1 (coefficient of x^2 understood to be 1), b is 0, and c is 2. For the purpose of emphasizing the values of a, b, and c with reference to the general form, this equation can be written

$$x^2 + 0x + 2 = 0$$

The coefficient of x^2 can never be 0; if it were 0, the equation would not be a quadratic. If the coefficients of x and x^0 are 0, then those terms do not normally appear. To say that the coefficient of x^0 is 0 is the same as saying that the constant term is 0 or is missing.

A ROOT of an equation in one variable is a value of the variable that satisfied the equation. Every equation in one variable, with constants as coefficients and positive integers as exponents, has as many roots as the exponent of the highest power. In other words, the number of roots is the same as the degree of the equation.

A fourth-degree equation has four roots, a cubic (third-degree) equation has three roots, a quadratic equation has two roots, and a linear equation has one root.

As an example, 6 and -1 are roots of the quadratic equation

$$x^2 - 5x - 6 = 0$$

This can be verified by substituting these values into the equation and noting that an identity results in each case.

Substituting x = 6 gives

$$6^2 - 5(6) - 6 = 0$$
$$36 - 36 = 0$$
$$0 = 0$$

Substituting x = -1 gives

$$(-1)^2 - 5(-1) - 6 = 0$$
$$1 + 5 - 6 = 0$$
$$6 - 6 = 0$$
$$0 = 0$$

Several methods of finding the roots of quadratic equations (SOLVING) are possible. The most common methods are solution by FACTORING and solution by the QUADRATIC FORMULA. Less commonly used methods of solution are accomplished by completing the square and by graphing.

SOLUTION BY FACTORING

The equation $x^2 - 36 = 0$ is a pure quadratic equation. There are two numbers which, when substituted for x, will satisfy the equation as follows:

$$(+6)^2 - 36 = 0$$
$$36 - 36 = 0$$

also

$$(-6)^2 - 36 = 0$$
$$36 - 36 = 0$$

Thus, +6 and -6 are roots of the equation

$$x^2 - 36 = 0$$

The most direct way to solve a pure quadratic (one in which no x term appears and the constant term is a perfect square) involves rewriting with the constant term in the right member, as follows:

$$x^2 = 36$$

Taking square roots on both sides, we have

$$x = \pm 6$$

The reason for expressing the solution as both plus and minus 6 is found in the fact that both +6 and -6, when squared, produce 36.

The equation

$$x^2 - 36 = 0$$

can also be solved by factoring, as follows:

$$x^2 - 36 = 0$$
$$(x + 6)(x - 6) = 0$$

We now have the product of two factors equal to zero. According to the zero factor law, if a product is zero, then one or more of its factors is zero. Therefore, at least one of the factors must be zero, and it makes no difference which one. We are free to set first one factor and then the other factor equal to zero. In so doing we derive two solutions or roots of the equation.

If x + 6 is the factor whose value is 0, then we have

$$x + 6 = 0$$
$$x = -6$$

If x - 6 is the zero factor, we have

$$x - 6 = 0$$
$$x = 6$$

When a three-term quadratic is put into simplest form, it is customary to place all the terms on the left side of the equality sign with the squared term first, the first-degree term next, and the constant term last, as in

$$9x^2 - 2x + 7 = 0$$

If the trinomial in the left member is readily factorable, the equation can be solved quickly by separating the trinominal into factors. Consider the equation

$$3x^2 - x - 2 = 0$$

By factoring the trinominal, the equation becomes

$$(3x + 2)(x - 1) = 0$$

Once again we have two factors, the product of which is 0. This means that one or the other of them (or both) must have the value 0. If the zero factor is $3x + 2$, we have

$$3x + 2 = 0$$
$$3x = -2$$
$$x = -\frac{2}{3}$$

If the zero factor is $x - 1$, we have

$$x - 1 = 0$$
$$x = 1$$

Substituting first $x = 1$ and then $x = -\frac{2}{3}$ in the original equation, we see that both roots satisfy it. Thus,

$$3(1)^2 - (1) - 2 = 0$$
$$3 - 1 - 2 = 0$$
$$0 = 0$$
$$3\left[-\frac{2}{3}\right]^2 - \left[-\frac{2}{3}\right] - 2 = 0$$
$$\frac{4}{3} + \frac{2}{3} - 2 = 0$$
$$0 = 0$$

In summation, when a quadratic may be readily factored, the process for finding its roots is as follows:

1. Arrange the equation in the order of the descending powers of the variable so that all the terms appear in the left member and zero appears in the right.
2. Factor the left member of the equation.
3. Set each factor containing the variable equal to zero and solve the resulting equations.
4. Check by substituting each of the derived roots in the original equation.

EXAMPLE: Solve the equation $x^2 - 4x = 12$ for x.

1. $x^2 - 4x - 12 = 0$
2. $(x - 6)(x + 2) = 0$
3. $x - 6 = 0$ $\qquad$ $x + 2 = 0$
   $x = 6$ $\qquad$ $x = -2$
4. $(6)^2 - 4(6) = 12$ $\quad (x = 6)$
   $36 - 24 = 12$
   $12 = 12$
   $(-2)^2 - 4(-2) = 12$ $\quad (x = -2)$
   $4 + 8 = 12$
   $12 = 12$

Practice problems. Solve the following equations by factoring:

1. $x^2 + 10x - 24 = 0$
2. $a^2 - a - 56 = 0$
3. $y^2 - 2y = 63$
4. $7y^2 - 19y - 6 = 0$
5. $m^2 - 4m = 96$

Answers:

1. $x = -12$
   $x = 2$
2. $a = 8$
   $a = -7$
3. $y = -7$
   $y = 9$
4. $y = 3$
   $y = -\frac{2}{7}$
5. $m = -8$
   $m = 12$

SOLUTION BY COMPLETING THE SQUARE

When a quadratic cannot be solved by factoring, or the factors are not readily seen, another method of finding the roots is needed. A method that may always be used for quadratics in one variable involves perfect square trinomials. These, we recall, are trinomials whose factors are identical. For example,

$$x^2 - 10x + 25 = (x - 5)(x - 5) = (x - 5)^2$$

Recall that in squaring a binomial, the third term of the resulting perfect square trinomial

is always the square of the second term of the binomial. The coefficient of the middle term of the trinomial is always twice the second term of the binomial. For example, when $(x + 4)$ is squared, we have

$$\begin{array}{l} x + 4 \\ \underline{x + 4} \\ x^2 + 4x \\ \quad\;\; \underline{+ 4x + 16} \\ x^2 + 8x + 16 \end{array}$$

Hence if both the second- and first-degree terms of a perfect square trinomial are known, the third may be written by squaring one-half the coefficient of the first-degree term.

Essentially, in completing the square, certain quantities are added to one member and subtracted from the other, and the equation is so arranged that the left member is a perfect square trinomial. The square roots of both members may then be taken, and the subsequent equalities may be solved for the variable.

For example,

$$x^2 + 5x - \frac{11}{4} = 0$$

cannot be readily factored. To solve for x by completing the square, we proceed as follows:

1. Leave only the second- and first-degree terms in the left member.

$$x^2 + 5x = \frac{11}{4}$$

(If the coefficient of x^2 is not 1, divide through by the coefficient of x^2.)

2. Complete the square by adding to both members the square of half the coefficient of the x term. In this example, one-half of the coefficient of the x term is $\frac{5}{2}$, and the square of $\frac{5}{2}$ is $\frac{25}{4}$. Thus,

$$x^2 + 5x + \frac{25}{4} = \frac{11}{4} + \frac{25}{4}$$

3. Factor the left member and simplify the right member.

$$\left(x + \frac{5}{2}\right)^2 = 9$$

4. Take the square root of both members.

$$\sqrt{\left(x + \frac{5}{2}\right)^2} = \pm\sqrt{9}$$

$$x + \frac{5}{2} = \pm 3$$

Remember that, in taking square roots on both sides of an equation, we must allow for the fact that two roots exist in every second-degree equation. Thus we designate both the plus and the minus root of 9 in this example.

5. Solve the resulting equations.

$$x + \frac{5}{2} = 3 \qquad\qquad x + \frac{5}{2} = -3$$

$$x = \frac{6}{2} - \frac{5}{2} \qquad\qquad x = -\frac{6}{2} - \frac{5}{2}$$

$$x = \frac{1}{2} \qquad\qquad x = -\frac{11}{2}$$

6. Check the results.

$$\left(\frac{1}{2}\right)^2 + \frac{5}{2} - \frac{11}{4} = 0$$

$$\frac{5}{2} - \frac{10}{4} = 0$$

$$0 = 0$$

$$\left(-\frac{11}{2}\right)^2 + (5)\left(-\frac{11}{2}\right) - \frac{11}{4} = 0$$

$$\frac{121}{4} - \frac{55}{2} - \frac{11}{4} = 0$$

$$\frac{110}{4} - \frac{55}{2} = 0$$

$$0 = 0$$

The process of completing the square may always be used to solve a quadratic equation. However, since this process may become complicated in more complex equations, a formula based on completing the square has been developed in which known quantities may be substituted in order to derive the roots of the quadratic equation. This formula is explained in the following paragraphs.

SOLUTION BY THE QUADRATIC FORMULA

The quadratic formula is derived by applying the process of completing the square to

solve for x in the general form of the quadratic equation, $ax^2 + bx + c = 0$. Remember that the general form represents every possible quadratic equation. Thus, if we can solve this equation for x, the solution will be in terms of a, b, and c. To solve this equation for x by completing the square, we proceed as follows:

1. Subtract the constant term, c, from both members.

$$ax^2 + bx + c = 0$$
$$ax^2 + bx = -c$$

2. Divide all terms by a so that the coefficient of the x^2 term becomes unity.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

3. Add the square of one-half the coefficient of the x term, $\frac{b}{a}$, to both members.

$$\text{Square } \frac{b}{2a}\text{: } \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

$$\text{Add: } x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

4. Factor the left member and simplify the right member.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

5. Take the square root of both members.

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

6. Solve for x.

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Thus, we have solved the equation representing every quadratic for its unknown in terms of its constants a, b, and c. Hence, in a given quadratic we need only substitute in the expression

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

the values of a, b, and c, as they appear in the particular equation, to derive the roots of that equation. This expression is called the QUADRATIC FORMULA. The general quadratic equation, $ax^2 + bx + c = 0$, and the quadratic formula should be memorized. Then, when a quadratic cannot be solved quickly by factoring, it may be solved at once by the formula.

EXAMPLE: Use the quadratic formula to solve the equation

$$x^2 + 30 - 11x = 0.$$

SOLUTION:

1. Set up the equation in standard form.

$$x^2 - 11x + 30 = 0$$

Then a (coefficient of x^2) = 1
b (coefficient of x) = -11
c (the constant term) = 30

2. Substituting,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-11) \pm \sqrt{(-11)^2 - 4(1)(30)}}{2(1)}$$
$$= \frac{11 \pm \sqrt{121 - 120}}{2}$$
$$= \frac{11 \pm 1}{2} = 6 \text{ or } 5$$

3. Checking:

When	When
$x = 6$,	$x = 5$,
$(6)^2 - 11(6) + 30 = 0$	$(5)^2 - 11(5) + 30 = 0$
$36 - 66 + 30 = 0$	$25 - 55 + 30 = 0$
$0 = 0$	$0 = 0$

EXAMPLE: Find the roots of

$$2x^2 - 3x - 1 = 0$$

Here, a = 2, b = -3, and c = -1.

Substituting into the quadratic formula gives

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-1)}}{2(2)}$$

$$= \frac{3 \pm \sqrt{9 + 8}}{4}$$

$$= \frac{3 \pm \sqrt{17}}{4}$$

The two roots are

$$x = \frac{3}{4} + \frac{1}{4}\sqrt{17} \text{ and } x = \frac{3}{4} - \frac{1}{4}\sqrt{17}$$

These roots are irrational numbers, since the radicals cannot be removed.

If the decimal values of the roots are desired, the value of the square root of 17 can be taken from appendix I of this course. Substituting $\sqrt{17} = 4.1231$ and simplifying gives

$$x_1 = \frac{3 + 4.1231}{4} \quad \text{and} \quad x_2 = \frac{3 - 4.1231}{4}$$

$$x_1 = \frac{7.1231}{4} \qquad x_2 = \frac{-1.1231}{4}$$

$$x_1 = 1.781 \qquad x_2 = -0.281$$

In decimal form, the roots of $2x^2 - 3x - 1 = 0$ to the nearest tenth are 1.8 and -0.3.

Notice that the subscripts, 1 and 2, are used to distinguish between the two roots of the equation. The three roots of a cubic equation in x might be designated x_1, x_2, and x_3. Sometimes the letter r is used for root. Using r, the roots of a cubic equation could be labeled r_1, r_2, and r_3.

Checking:

When $x_1 = \frac{3 + \sqrt{17}}{4}$

$$2x^2 - 3x - 1 = 0$$

then

$$2\left(\frac{3 + \sqrt{17}}{4}\right)^2 - 3\left(\frac{3 + \sqrt{17}}{4}\right) - 1 = 0$$

$$\frac{(3 + \sqrt{17})^2}{8} - \frac{9 + 3\sqrt{17}}{4} - 1 = 0$$

$$\frac{9 + 6\sqrt{17} + 17 - 18 - 6\sqrt{17} - 8}{8} = 0$$

$$0 = 0$$

When

$$x_2 = \frac{3 - \sqrt{17}}{4}$$

then

$$2\left(\frac{3 - \sqrt{17}}{4}\right)^2 - 3\left(\frac{3 - \sqrt{17}}{4}\right) - 1 = 0$$

$$\frac{9 - 6\sqrt{17} + 17}{8} - \frac{9 - 3\sqrt{17}}{4} - 1 = 0$$

Multiplying both members of the equation by 8, the LCD, we have

$$8\left(\frac{9 - 6\sqrt{17} + 17}{8}\right) - 8\left(\frac{9 - 3\sqrt{17}}{4}\right) - 8(1) = 0$$

$$9 - 6\sqrt{17} + 17 - 2(9 - 3\sqrt{17}) - 8 = 0$$

$$9 - 6\sqrt{17} + 17 - 18 + 6\sqrt{17} - 8 = 0$$

$$0 = 0$$

Practice problems. Use the quadratic formula to find the roots of the following equations:

1. $3x^2 - 20 - 7x = 0$
2. $4x^2 - 3x - 5 = 0$
3. $15x^2 - 22x - 5 = 0$
4. $x^2 + 7x = 8$

Answers:

1. $x_1 = 4$, $x_2 = -\frac{5}{3}$
2. $x_1 = \frac{3 + \sqrt{89}}{8}$, $x_2 = \frac{3 - \sqrt{89}}{8}$
3. $x_1 = \frac{5}{3}$, $x_2 = -\frac{1}{5}$
4. $x_1 = 1$, $x_2 = -8$

GRAPHICAL SOLUTION

A fourth method of solving a quadratic equation is by means of graphing. In graphing linear equations using both axes as reference, we recall that an independent variable, x, and a dependent variable, y, were needed. The coordinates of points on the graph of the equation were designated (x, y).

Since the quadratics we are considering contain only one variable, as in the equation

$$x^2 - 8x + 12 = 0$$

we cannot plot values for the equations in the present form using both x and y axes. A dependent variable, y, is necessary.

If we think of the expression

$$x^2 - 8x + 12$$

as a function, then this function can be considered to have many possible numerical values, depending on what value we assign to x. The particular value or values of x which cause the value of the function to be 0 are solutions for the equation

$$x^2 - 8x + 12 = 0$$

For convenience, we may choose to let y represent the function

$$x^2 - 8x + 12$$

If numerical values are now assigned to x, the corresponding values of y may be calculated. When these pairs of corresponding values of x and y are tabulated, the resulting table provides the information necessary for plotting a graph of the function.

EXAMPLE: Graph the equation

$$x^2 + 2x - 8 = 0$$

and from the graph write the roots of the equation.

SOLUTION:

1. Let $y = x^2 + 2x - 8$.

2. Make a table of the y values corresponding to the value assigned x, as shown in table 16-1.

Table 16-1.—Tabulation of x and y values for the function $y = x^2 + 2x - 8$.

if x = -----	-5	-4	-3	-2	-1	0	1	2	3
then y ---	7	0	-5	-8	-9	-8	-5	0	7

3. Plot the pairs of x and y values that appear in the table as coordinates of points on a rectangular coordinate system as in figure 16-1 (A).

4. Draw a smooth curve through these points, as shown in figure 16-1 (B).

Notice that this curve crosses the X axis in two places. We also recall that, for any point on the X axis, the y coordinate is zero. Thus, in the figure we see that when y is zero, x is -4 or +2. When y is zero, furthermore, we have the original equation,

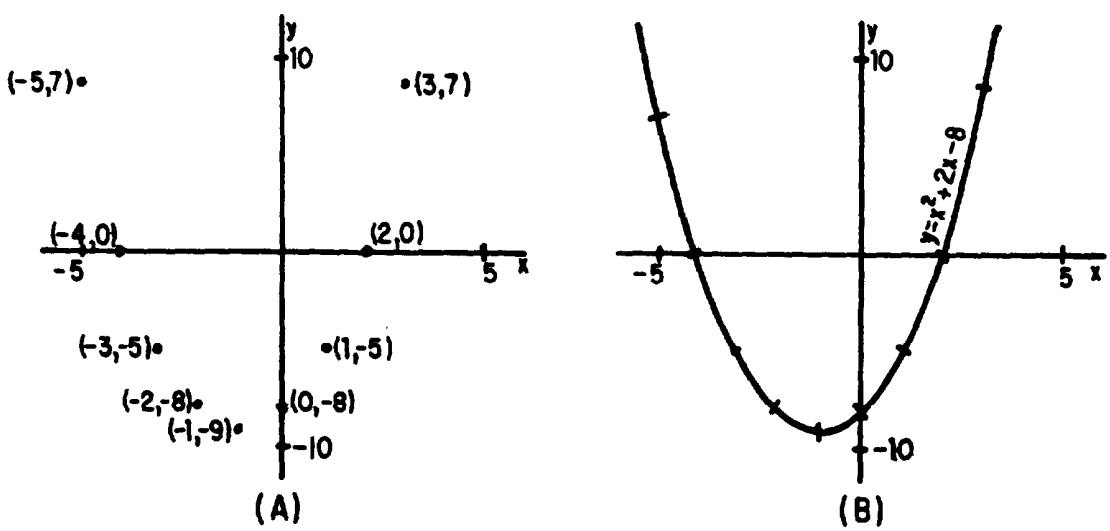

Figure 16-1.—Graph of the equation $y = x^2 + 2x - 8$. (A) Points plotted; (B) curve drawn through plotted points.

$$x^2 + 2x - 8 = 0$$

Thus, the values of x at these points where the graph of the equation crosses the X axis ($x = -4$ or $+2$) are solutions to the original equation. We may check these results by solving the equation algebraically. Thus,

$$x^2 + 2x - 8 = 0$$

$$(x + 4)(x - 2) = 0$$

$$x_1 + 4 = 0 \qquad x_2 - 2 = 0$$

$$x_1 = -4 \qquad x_2 = 2$$

Check:

$$(-4)^2 + 2(-4) - 8 = 0 \qquad (2)^2 + 2(2) - 8 = 0$$

$$16 - 8 - 8 = 0 \qquad 4 + 4 - 8 = 0$$

$$0 = 0 \qquad 0 = 0$$

The curve in figure 16-1 (B) is called a PARABOLA. Every quadratic of the form $ax^2 + bx + c = y$ will have a graph of this general shape. The curve will open downward if a is negative, and upward if a is positive.

Graphing provides a fourth method of finding the roots of a quadratic in one variable. When the equation is graphed, the roots will be the X intercepts (those values of x where the curve crosses the X axis). The X intercepts are the points at which y is 0.

Practice problems. Graph the following quadratic equations and read the roots of each equation from its graph

1. $x^2 - 4x - 8 = 0$
2. $6x - 5 - x^2 = 0$

Answers:

1. See figure 16-2. $x = 5.5$; $x = -1.5$
2. See figure 16-3. $x = 1$; $x = 5$

MAXIMUM AND MINIMUM POINTS

It will be seen from the graphs of quadratics in one variable that a parabola has a maximum or minimum value, depending on whether the curve opens upward or downward. Thus, when a is negative the curve passes through a maximum value; and when a is positive, the curve passes through a minimum value. Often these maximum or minimum values comprise the only information needed for a particular problem.

In higher mathematics it can be shown that the X coordinate, or abscissa, of the maximum or minimum value is

$$x = \frac{-b}{2a}$$

In other words, if we divide minus the coefficient of the x term by twice the coefficient of the x^2 term, we have the X coordinate of the maximum or minimum point. If we substitute this value for x in the original equation, the result is the Y value or ordinate, which corresponds to the X value.

For example, we know that the graph of the equation

$$x^2 + 2x - 8 = y$$

passes through a minimum value because a is positive. To find the coordinates of the point where the parabola has its minimum value, we note that $a = 1$, $b = 2$, $c = -8$. From the rule given above, the X value of the minimum point is

$$x = \frac{-b}{2a}$$

$$x = \frac{-(2)}{2(1)}$$

$$x = -1$$

Substituting this value for x in the original equation, we have the value of the Y coordinate of the minimum point. Thus,

$$(-1)^2 + 2(-1) - 8 = y$$

$$1 - 2 - 8 = y$$

$$-9 = y$$

The minimum point is (-1, -9). From the graph in figure 16-1 (A), we see that these coordinates are correct. Thus, we can quickly and easily find the coordinates of the minimum or maximum point for any quadratic of the form $ax^2 + bx + c = 0$.

Practice problems. Without graphing, find the coordinates of the maximum or minimum points for the following equations and state whether they are maximum or minimum.

1. $2x^2 - 5x + 2 = 0$
2. $68 - 3x - x^2 = 0$
3. $3 + 7x - 6x^2 = 0$
4. $24x^2 - 14x = 3$

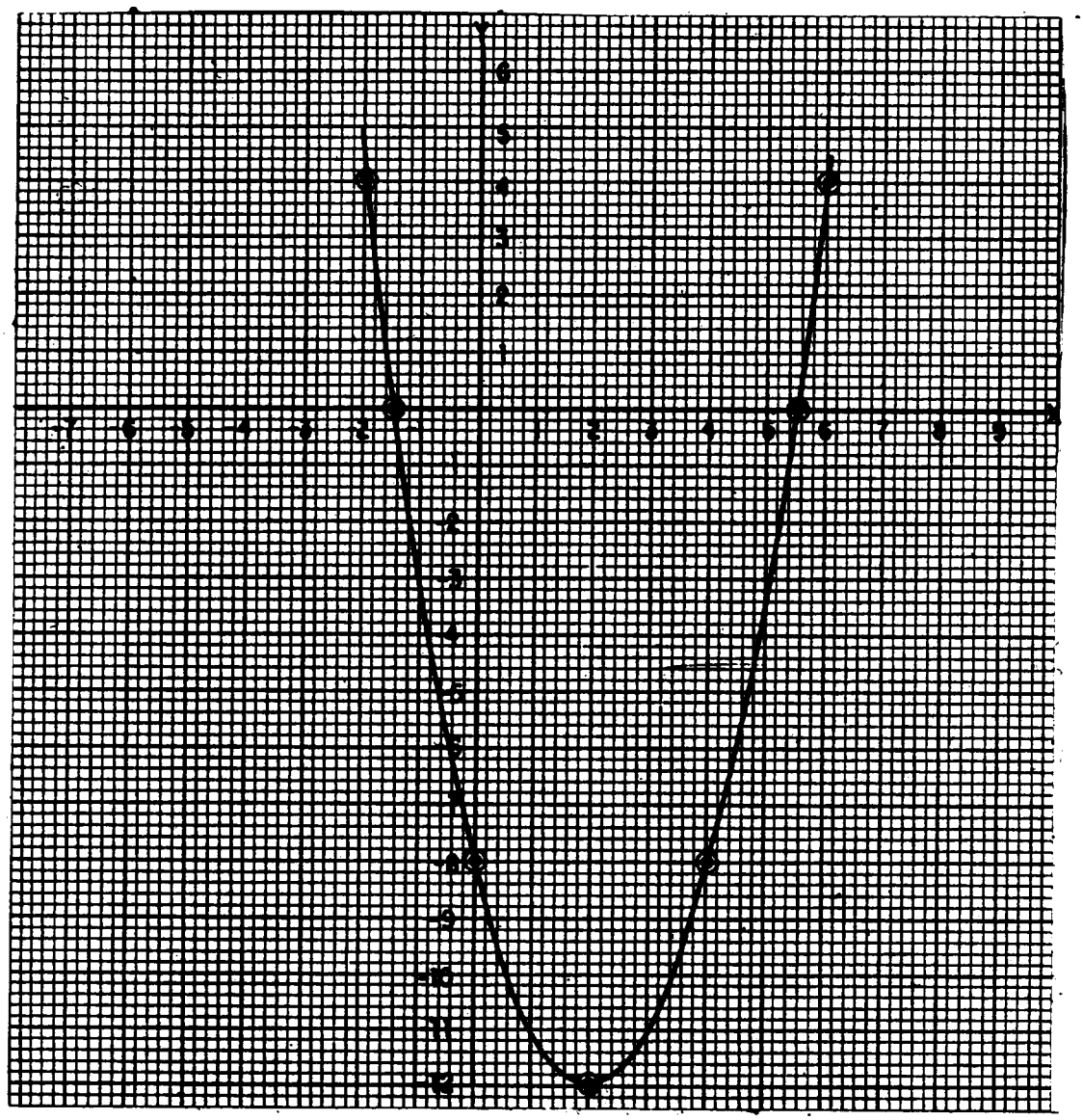

Figure 16-2.—Graph of $x^2 - 4x - 8 = 0$.

Answers:

1. $x = \frac{5}{4}$ Minimum

   $y = -\frac{9}{8}$

2. $x = -\frac{3}{2}$ Maximum

   $y = \frac{281}{4}$

3. $x = \frac{7}{12}$ Maximum

   $y = \frac{121}{24}$

4. $x = \frac{7}{24}$ Minimum

   $y = -\frac{121}{24}$

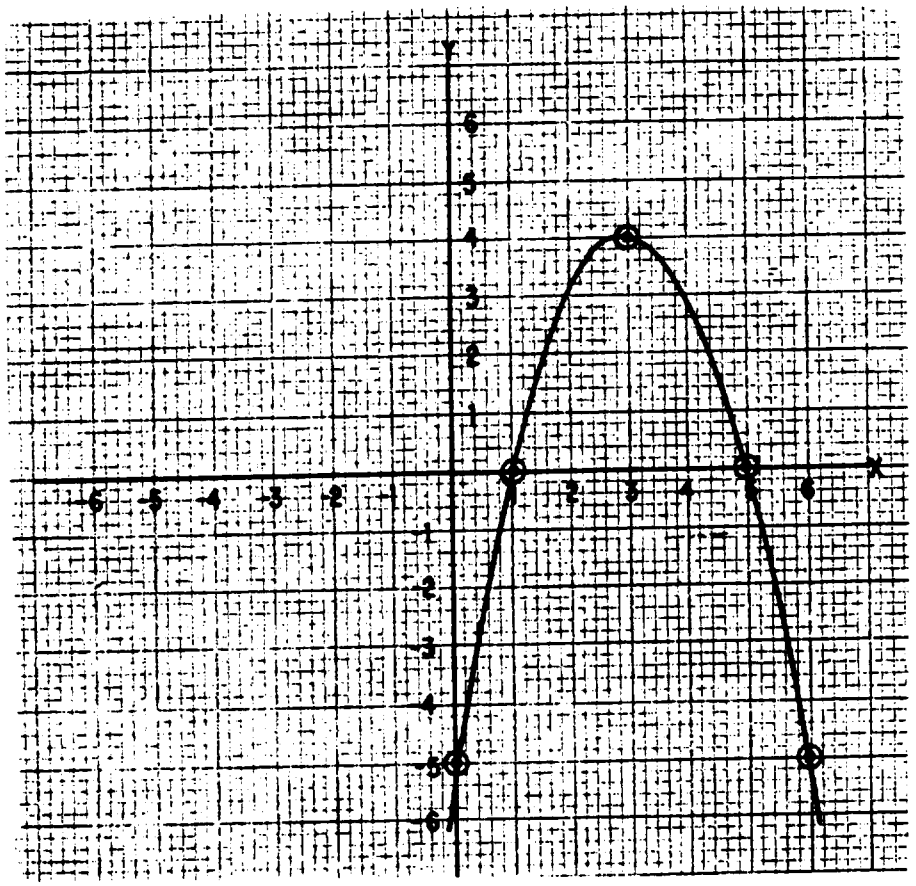

Figure 16-3.—Graph of $6x - 5 - x^2 = 0$.

THE DISCRIMINANT

The roots of a quadratic equation may be classified in accordance with the following criteria:

1. Real or imaginary.
2. Rational or irrational.
3. Equal or unequal.

The task of discriminating among these possible characteristics to find the nature of the roots is best accomplished with the aid of the quadratic formula. The part of the quadratic formula which is used is called the DISCRIMINANT.

If the roots of a quadratic are denoted by the symbols r_1 and r_2, then the following relations may be stated:

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

We can show that the character of the roots is dependent upon the form taken by the expression

$$b^2 - 4ac$$

which is the quantity under the radical in the formula. This expression is the DISCRIMINANT of a quadratic equation.

IMAGINARY ROOTS

Since there is a radical in each root, there is a possibility that the roots could be imaginary. They are imaginary when the number under the radical in the quadratic formula is negative (less than 0). In other words, when the value of the discriminant is less than 0, the roots are imaginary.

EXAMPLE:

$$x^2 + x + 1 = 0$$

$$a = 1,\ b = 1,\ c = 1$$

$$\begin{aligned} b^2 - 4ac &= (1)^2 - 4(1)(1) \\ &= 1 - 4 \\ &= -3 \end{aligned}$$

Thus, without further work, we know that the roots are imaginary.

CHECK: The roots are

$$r_1 = \frac{-1 + \sqrt{-3}}{2} \qquad r_2 = \frac{-1 - \sqrt{-3}}{2}$$

$$r_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2} \qquad r_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

We recognize both of these numbers as being imaginary.

We may also conclude that when one root is imaginary the other will also be imaginary. This is because the pairs of imaginary roots are always conjugate complex numbers. If one root is of the form $a + ib$, then $a - ib$ is also a root. Knowing that imaginary roots always occur in pairs, we can conclude that a quadratic equation always has either two imaginary roots or two real roots.

Practice problems. Using the discriminant, state whether the roots of the following equations are real or imaginary:

1. $x^2 - 6x - 16 = 0$
2. $x^2 - 6x = -12$
3. $3x^2 - 10x + 50 = 0$
4. $6x^2 + x = 1$

Answers:

1. Real
2. Imaginary
3. Imaginary
4. Real

EQUAL OR DOUBLE ROOTS

If the discriminant $b^2 - 4ac$ equals zero, the radical in the quadratic formula becomes zero. In this case the roots are equal; such roots are sometimes called double roots.

Consider the equation

$$9x^2 + 12x + 4 = 0$$

Comparing with the general quadratic, we notice that

$$a = 9,\ b = 12, \text{ and } c = 4$$

The discriminant is

$$\begin{aligned} b^2 - 4ac &= 12^2 - 4(9)(4) \\ &= 144 - 144 \\ &= 0 \end{aligned}$$

Therefore, the roots are equal.

CHECK: From the formula

$$r_1 = \frac{-12 + 0}{2(9)} \qquad r_2 = \frac{-12 - 0}{2(9)}$$

$$r_1 = -\frac{2}{3} \qquad r_2 = -\frac{2}{3}$$

The equality of the roots is thus verified.

The roots can be equal only if the trinomial is a perfect square. Its factors are equal. Factoring the trinomial in

$$9x^2 + 12x + 4 = 0$$

we see that

$$(3x + 2)^2 = 0$$

Since the factor $3x + 2$ is squared, we actually have

$$3x + 2 = 0$$

twice, and we have

$$x = -\frac{2}{3}$$

twice.

The fact that the same root must be counted twice explains the use of the term "double root." A double root of a quadratic equation is always rational because a double root can occur only when the radical vanishes.

REAL AND UNEQUAL ROOTS

When the discriminant is positive, the roots must be real. Also they must be unequal since equal roots occur only when the discriminant is zero.

Rational Roots

If the discriminant is a perfect square, the roots are rational. For example, consider the equation

$$3x^2 - x - 2 = 0$$

in which

$$a = 3,\ b = -1,\ \text{and}\ c = -2$$

The discriminant is

$$\begin{aligned} b^2 - 4ac &= (-1)^2 - 4(3)(-2) \\ &= 1 + 24 \\ &= 25 \end{aligned}$$

We see that the discriminant, 25, is a perfect square. The perfect square indicates that the radical in the quadratic formula can be removed, that the roots of the equation are rational, and that the trinomial can be factored. In other words, when we evaluate the discriminant and find it to be a perfect square, we know that the trinomial can be factored.

Thus,

$$3x^2 - x - 2 = 0$$
$$(3x + 2)(x - 1) = 0$$

from which

$$3x + 2 = 0 \qquad x - 1 = 0$$
$$x = -\frac{2}{3} \qquad x = 1$$

We see that the information derived from the discriminant is correct. The roots are real, unequal, and rational.

Irrational Roots

If the discriminant is not a perfect square, the radical cannot be removed and the roots are irrational.

Consider the equation

$$2x^2 - 4x + 1 = 0$$

in which

$$a = 2,\ b = -4,\ \text{and}\ c = 1.$$

The discriminant is

$$\begin{aligned} b^2 - 4ac &= (-4)^2 - 4(2)(1) \\ &= 16 - 8 \\ &= 8 \end{aligned}$$

This discriminant is positive and not a perfect square. Thus the roots are real, unequal, and irrational.

To check the correctness of this information, we derive the roots by means of the formula. Thus,

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{4 \pm \sqrt{8}}{4} \\ &= \frac{2 \pm \sqrt{2}}{2} \end{aligned}$$

$$x = 1 + \frac{\sqrt{2}}{2} \text{ or } x = 1 - \frac{\sqrt{2}}{2}$$

This verifies the conclusions reached in evaluating the discriminant. When the discriminant is a positive number, not a perfect square, it is useless to attempt to factor the trinomial. The formula is needed to find the roots. They will be real, unequal, and irrational.

SUMMARY

The foregoing information concerning the discriminant may be summed up in the following four rules:

1. If $b^2 - 4ac$ is a perfect square or zero, the roots are rational; otherwise they are irrational.
2. If $b^2 - 4ac$ is negative (less than zero), the roots are imaginary.
3. If $b^2 - 4ac$ is zero, the roots are real, equal, and rational.
4. If $b^2 - 4ac$ is greater than zero, the roots are real and unequal.

Practice problems. Determine the character of the roots of each of the following equations:

1. $x^2 - 7x + 12 = 0$
2. $9x^2 - 6x + 1 = 0$
3. $2x^2 - x + 1 = 0$
4. $2x - 2x^2 + 6 = 0$

Answers:

1. Real, unequal, rational
2. Real, equal, rational
3. Imaginary
4. Real, unequal, irrational

GRAPHICAL INTERPRETATION OF ROOTS

When a quadratic is set equal to y and the resulting equation is graphed, the graph will reveal the character of the roots, but it may not reveal whether the roots are rational or irrational.

Consider the following equations:

1. $x^2 + 6x - 3 = y$
2. $x^2 + 6x + 9 = y$
3. $x^2 + 6x + 13 = y$

The graphs representing these equations are shown in figure 16-4.

We recall that the roots of the equation are the values of x at those points where y is zero. Y is zero on the graph anywhere along the X axis. Thus, the roots of the equation are the positions where the graph crosses the X axis. In parabola No. 1 (fig. 16-4) we see immediately that there are two roots to the equation and that they are unequal. These roots appear to be -6.5 and 0.5. Algebraically, we find them to be the irrational numbers

$$-3 + 2\sqrt{3} \text{ and } -3 - 2\sqrt{3}.$$

For equation No. 2 (fig. 16-4), the parabola just touches the X axis at x = -3. This means that both roots of the equation are the same—that is, the root is a double root. At the point where the parabola touches the X axis, the two roots of the quadratic equation have moved

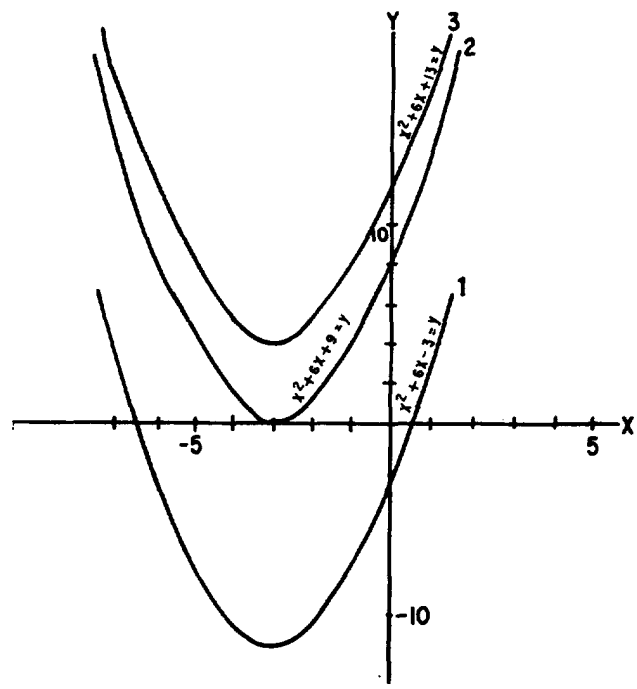

Figure 16-4.—Graphical interpretation of roots.

together and the two points of intersection of the parabola and the X axis are coincident. The quantity -3 as a double root agrees with the algebraic solution.

When the equation No. 3 (fig. 16-4) is solved algebraically, we see that the roots are -3 + 2i and -3 - 2i. Thus they are imaginary. Parabola No. 3 does not cross the X axis. When this situation occurs, imaginary roots are implied. Only equations having real roots will have graphs that cross or touch the X axis. Thus we may determine from the graph of an equation whether the roots are real or imaginary.

VERBAL PROBLEMS INVOLVING QUADRATIC EQUATIONS

Many practical problems give rise to quadratic equations. In such problems it often happens that one of the roots will have no meaning. We must select the root that satisfies the conditions of the problem.

Consider the following example: The length of a plot of ground exceeds its width by 7 ft and the area of the plot is 120 sq ft. What are the dimensions?

SOLUTION:

Let x = length

y = width

then

$$x - y = 7 \qquad (1)$$

and

$$xy = 120 \qquad (2)$$

Solving (1) for y, $y = x - 7$

Substituting $(x - 7)$ for y in (2)

$$x(x - 7) = 120$$

Therefore

$$x^2 - 7x - 120 = 0$$

$$(x - 15)(x + 8) = 0$$

$$x = 15, \qquad x = -8$$

Thus, length = +15 or -8.

But the length obviously cannot be a negative value. Therefore, we reject -8 as a value for x and use only the positive value, +15. Then from equation (1),

$$15 - y = 7$$

$$y = 8$$

Length = 15, Width = 8

Practice problems. Solve the following problems by forming quadratic equations:

1. A rectangular plot is 8 yd by 24 yd. If the length and width are increased by the same amount, the area is increased by 144 sq yd. How much is each dimension increased?

2. Two cars travel at uniform rates of speed over the same route a distance of 180 mi. One goes 5 mph slower than the other and takes 1/2 hr longer to make the run. How fast does each car travel?

Answers:

1. Length and width are each increased by 4 yd.

2. Faster car: 45 mph.
   Slower car: 40 mph.